

SKEIN THEORY FOR THE ADE PLANAR ALGEBRAS

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ABSTRACT. We give generators and relations for the planar algebras corresponding to ADE subfactors. We also give a basis and an algorithm to express an arbitrary diagram as a linear combination of these basis diagrams.

1. INTRODUCTION

The notion of a planar algebra is due to Jones [Jon99]. The roughly equivalent notion of a spider is due to Kuperberg [Kup96]. Planar algebras arise in many contexts where there is a reasonably nice category with tensor products and duals. Examples are the category of representations of a quantum group, or of bimodules coming from a subfactor.

The subfactor algebras of index less than 4 can be classified into the two infinite families A_N and D_{2N} , and the two sporadic examples E_6 and E_8 . See [GHJ89], [Ocn88], [Izu91], and [KO02] for the story of this “ADE” classification.

The *Kuperberg program* can be summarized as follows.

Give a presentation for every interesting planar algebra, and prove as much as possible about the planar algebra using only its presentation.

The planar algebras corresponding to subfactors of type A_N are fairly well understood. In [MPS08], Morrison, Peters and Snyder basically complete the Kuperberg program in the D_{2N} case.

The aim of this paper is to extend the results of [MPS08] types E_6 and E_8 . However our approach is different. Whereas [MPS08] use only combinatorial arguments starting from their presentation, we will rely on the existence of the subfactor planar algebra and some of its known properties. Thus this paper is not completely in the spirit of the Kuperberg program. As a compensation, we will address the following program, suggested by Jones in Appendix B of [Jon03].

Give a basis for every interesting planar algebra, and an algorithm to express any given diagram as a linear combination of basis elements.

Most of this paper concerns the subfactor of type E_8 . In Section 3 we define two planar algebras. The planar algebra \mathcal{P} is defined by a presentation in terms of one generator and five relations. The planar algebra \mathcal{P}' is the subfactor planar algebra whose principal graph is the E_8 graph. We give the properties of \mathcal{P}' that we need. In Section 4 we prove that \mathcal{P} is isomorphic to \mathcal{P}' . In Section 5 we define a set of diagrams that will form a basis for our planar algebra. The proof that the basis spans is constructive, although we have not tried to give an efficient algorithm.

Finally, in Section 6, we explain how our methods can be applied to types E_6 , A_N and D_{2N} .

2. PLANAR ALGEBRAS

We give a brief and impressionistic review of the definition of a planar algebra. For the details, see the preprint [Jon99] at Vaughan Jones' website.

Something that is called an “algebra” is usually a vector space together with one or more additional operations. A planar algebra \mathcal{P} consists of infinitely many vector spaces (or one graded vector space if you prefer), together with infinitely many operations. For every non-negative integer k , we have a vector space \mathcal{P}_k . For every planar arc diagram T , we have a multilinear n -ary operation

$$\mathcal{P}(T): \mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_n} \rightarrow \mathcal{P}_{k_0},$$

where n is the number of internal disks in T , k_1, \dots, k_n are the numbers of ends of strands on the internal disks of T , and k_0 is the number of ends of strands on the external disk.

In practice, \mathcal{P}_k will be spanned by some kind of diagrams in a disk. Each diagram in \mathcal{P}_k has k endpoints of strands on its boundary. The action of a planar arc diagram T is given by gluing diagrams into the interior disks of T , matching up endpoints of strands on the boundary of the diagrams with the endpoints of strands in T . To determine “which way up” to glue the diagrams, we use basepoints on the boundaries of diagrams and on the boundaries of input disks of T . These basepoints are indicated by a star, and are never allowed to coincide with the endpoints of strands.

Note that, for ease of exposition, we only work with “unshaded” planar algebras.

2.1. Composition. It is often convenient to draw an element of a planar algebra in a rectangle instead of a round disk. When we do this, the basepoint will always be at the top left corner, and the endpoints of strands will be on the top and bottom edges.

Let \mathcal{P}_b^a denote the elements of \mathcal{P}_{a+b} , drawn in a rectangle, with a endpoints at the top and b at the bottom. If $A \in \mathcal{P}_b^a$ and $B \in \mathcal{P}_c^b$ then the *composition* of A and B is the element $AB \in \mathcal{P}_c^a$ obtained by stacking A on top of B . Note that the meaning of this composition depends on the value of b , which must be made clear from context. (Here, we are blurring the distinction between the planar algebra and the corresponding category, as defined in [MPS08].)

2.2. Quantum integers. Suppose q is a non-zero complex number. The *quantum integer* $[n]$ is given by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

These appear only briefly in this paper, and they can be treated as constants whose precise value is unimportant. However they play an important role behind the scenes, for example in the definition of the Jones-Wenzl idempotents, and in the properties of the subfactor planar algebra.

2.3. Temperley-Lieb planar algebra. A *Temperley-Lieb diagram* is a finite collection of disjoint properly embedded edges in a disk, together with a basepoint on the boundary. These form a planar algebra as follows. Suppose T is a planar arc diagram with n holes, and D_1, \dots, D_n are Temperley-Lieb diagrams with the appropriate number of endpoints. We can create a new Temperley-Lieb diagram A by inserting D_1, \dots, D_n into the holes in T , and deleting any resulting strands that form closed loops. Let m be the number of closed loops that were deleted. Then T maps the n -tuple D_1, \dots, D_n to $[2]^m A$.

The planar algebra of Temperley-Lieb diagrams is called the *Temperley-Lieb planar algebra*, and will be denoted \mathcal{TL} . It can be defined more briefly as the planar algebra with no generators and a single defining relation

$$\bigcirc = [2] \bigcirc.$$

Every planar algebra we consider will satisfy the above relation, and hence contain an image of \mathcal{TL} .

We now list some important examples of Temperley-Lieb diagrams. The *identity diagram* $\text{id}_n \in \mathcal{TL}_n^n$ is the diagram consisting of n vertical strands in a rectangle.

Suppose $D \in \mathcal{TL}_n^m$ is a Temperley-Lieb diagram drawn in a rectangle. We will say D *contains a cup* if it contains a strand that has both endpoints on the top edge of the rectangle. We say D *is a cup* if it consists of n vertical strands and one strand that has both endpoints on the top of the rectangle.

Similarly, a *cap* is a diagram in \mathcal{TL}_{n+2}^n that has n vertical strands and one strand with both endpoints on the bottom of the rectangle.

The *Jones-Wenzl idempotent* P_n is unique element of \mathcal{TL}_n^n with the following properties.

- When P_n is expressed as a linear combination of Temperley-Lieb diagrams, the diagram id_n occurs with coefficient 1.
- If $X \in \mathcal{TL}_n^{n-2}$ is any cap then $XP_n = 0$.
- If $Y \in \mathcal{TL}_{n-2}^n$ is any cup then $P_nY = 0$.

In all of our examples, q will be of the form $e^{i\pi/N}$. In this case the element P_n exists and is unique for all $n \leq N-1$, but does not exist for $n \geq N$.

Let a *crossing* be the following element of \mathcal{P}_4 .

$$\bigcirc \times = iq^{\frac{1}{2}} \bigcirc \left(\begin{array}{c} \text{cap} \end{array} \right) - iq^{-\frac{1}{2}} \bigcirc \left(\begin{array}{c} \text{cup} \end{array} \right)$$

The crossing allows us to consider knot and tangle diagrams as representing elements of the Temperley-Lieb planar algebra. We can express a diagram with k crossings as a linear combination of 2^k diagrams that have no crossings. This process is called *resolving the crossings*.

The crossing satisfies Reidemeister moves two and three. In place of Reidemeister one, we have the following.

$$\bigcirc \left(\begin{array}{c} \text{strand} \end{array} \right) = iq^{\frac{3}{2}} \left(\begin{array}{c} \text{strand} \end{array} \right)$$

3. THE E_8 PLANAR ALGEBRA

The purpose of this section is to define the planar algebras \mathcal{P} and \mathcal{P}' . The first is given by a presentation, and the second is the subfactor planar algebra whose principal graph is the E_8 graph. In the next section, we will show that \mathcal{P} and \mathcal{P}' are isomorphic.

3.1. The presentation. We define \mathcal{P} in terms of generators and relations. There is just one generator $S \in \mathcal{P}_{10}$.

Before we list the relations, we define some notation. Let $q = e^{i\pi/30}$. Let $\omega = e^{6i\pi/5}$. Let $\rho(S)$, $\tau(S)$, S^2 and \hat{S} be as shown in Figure 1. We call $\rho(S)$ the *rotation* of S and $\tau(S)$ the *partial trace* of S .

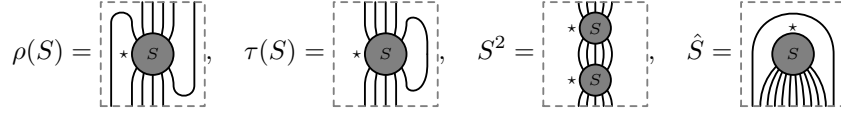
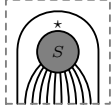
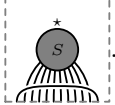


FIGURE 1. The diagrams $\rho(S)$, $\tau(S)$, S^2 and \hat{S} , respectively

The defining relations of \mathcal{P} are as follows.

- $\bigcirc = [2] \bigcirc$,
- $\rho(S) = \omega S$,
- $\tau(S) = 0$,
- $S^2 = S + [2]^2 [3] P_5$,
- $\hat{S} P_{12} = 0$.

We call the first four relations the *bubble bursting*, *chirality*, *partial trace*, and *quadratic* relation, respectively. The fifth relation is equivalent to the following *braiding relation*.

Lemma 3.1.  = .

Proof. Let X denote the diagram on the right hand side of the braiding relation. We must show that $\hat{S} = X$.

Suppose Y is the cup

$$Y = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \text{V} \\ \hline \end{array}$$

Then

$$\hat{S}Y = \rho(S) = \omega S.$$

To compute XY , first apply Reidemeister one to the rightmost crossing, giving a factor of $iq^{3/2}$. Now resolve the remaining nine crossings. All but one of the resulting terms contains a cup connected directly to S , so can be eliminated. The easiest way to see this is to resolve the crossings one by one, working from right to left. In the end, we obtain

$$XY = (iq^{3/2})(iq^{1/2})^9 S = -q^6 S.$$

By comparing the above expressions, we find that $\hat{S}Y = XY$.

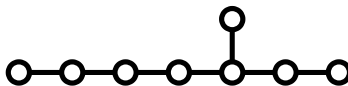
$$Y = \begin{bmatrix} - & | & | & | & | \\ \hline 0 & | & | & | & | \\ \hline \end{bmatrix}$$

Recall that P_{12} is a linear combination of Temperley-Lieb diagrams. Every term in this linear combination contains a cup, except for the identity, which occurs with coefficient one. Thus

$$\hat{S} = X \Leftrightarrow \hat{S}P_{12} = XP_{12}.$$

But XP_{12} is clearly zero. Thus the desired result $\hat{S} = X$ is equivalent to the relation $\hat{S}P_{12} = 0$. \square

3.2. The subfactor planar algebra. The E_8 graph is as follows.



We now give some known properties of \mathcal{P}' . The proofs require some background knowledge on subfactor planar algebras and principal graphs. The meaning of these terms is explained in [MPS08] and [Jon99].

Proof. We take R to be a generator for the space of morphisms from P_{10} to the empty diagram.

The chirality relation is given in [Jon99, Theorem 4.2.13]. The “chirality” in that theorem is the value of ω^2 in our chirality relation. The shaded planar algebra is only unique up to complex conjugation. The unshaded version also requires an arbitrary choice of sign of ω . We chose a value of ω so that the braiding relation hold with our definition of a crossing.

The quadratic relation is equation 4.3.5 in [Jon03]. There is currently a misprint: it should read $R^2 = (1 - r)R + rp_n$. However the proof is correct. In our context, $r = [2]^2/[3]$, and we have rescaled R by a factor of $[3]$.

The relation $\hat{S}P_{12} = 0$ follows from the fact that there is no non-zero morphism from P_{12} to the empty diagram in \mathcal{P}' . \square

Lemma 3.3. *In \mathcal{P}' , id_7 is equal to a linear combination of diagrams of the form AB such that $A \in (\mathcal{P}')_m^7$, and $B \in (\mathcal{P}')_7^m$, for some $m < 7$.*

Proof. This comes down to the fact that the E_8 graph has diameter less than seven. Every minimal idempotent is a summand of id_m for some $m < 7$. \square

Lemma 3.4. $P_{29} = 0$ in \mathcal{P}' .

Proof. This holds in any subfactor planar algebra where $[30] = 0$. \square

We now give the dimension of \mathcal{P}'_n in a form that will be useful later.

Definition. Suppose $Y \in \mathcal{TL}_n^m$ is a Temperley-Lieb diagram drawn in a rectangle. Let x_0, \dots, x_n be a sequence of points on the bottom edge of Y , ordered from left to right, occupying the $n + 1$ spaces between the endpoints of strands. We say Y is *JW-reduced* if Y contains no cups, and for all $i = 0, \dots, n$, there are at most 28 strands that have one endpoint to the left of x_i and the other endpoint either to the right of x_i or on the top edge of Y .

Lemma 3.5. For $m, n \geq 0$, let d_n^m be the number of JW-reduced diagrams in \mathcal{TL}_n^m . Then $\dim(\mathcal{P}'_n) = d_n^0 + d_n^{10} + d_n^{18} + d_n^{28}$.

Proof. The idea is to decompose id_n into a direct sum of minimal idempotents in the category corresponding to \mathcal{P}' . The dimension of \mathcal{P}'_n is the number of copies of the empty diagram in this decomposition.

First we work in the image of the Temperley-Lieb algebra in \mathcal{P}' . This is the Temperley-Lieb algebra satisfying the bubble bursting relation and $P_{29} = 0$. The JW-reduced diagrams in \mathcal{TL}_n^m give a basis for the space of morphisms from id_n to P_m . Thus d_n^m is the number of copies of P_m in the decomposition of id_n .

Now decompose P_m into minimal idempotents, working in \mathcal{P}' . This is easy to do using the information encoded in the principal graph. The number of copies of the empty diagram in the decomposition of P_m is one if $m \in \{0, 10, 18, 28\}$ and zero otherwise.

Combining these two facts, we see that the number of copies of the empty diagram in the decomposition of id_n into minimal idempotents is as claimed. \square

4. THE ISOMORPHISM

The aim of this section is to prove the following.

Theorem 4.1. \mathcal{P}' is isomorphic to \mathcal{P}

By Lemma 3.2, there is a surjective planar algebra morphism Φ from \mathcal{P} to \mathcal{P}' , taking S to R . It remains to show that Φ is injective.

Lemma 4.2. Every element of \mathcal{P}_0 is a scalar multiple of the empty diagram.

Proof. Suppose D is a diagram in \mathcal{P}_0 . We must show that D is a scalar multiple of the empty diagram. Let m be the number of copies of S in D . We will use induction on m . By the braiding relation, we can assume the copies of S lie on the vertices of a regular m -gon, and that every strand lies inside this m -gon.

By resolving all crossings, we can reduce to the case that D is a diagram with no crossings. By the bubble bursting relation, we can assume D contains no closed loops. By the chirality and partial trace relations, we can assume D contains no strand with both endpoints on the same copy of S . Thus every strand in D connects two distinct copies of S .

We can think of D as a triangulated m -gon, where the edges have multiplicities. (We may need to add some edges with multiplicity zero if we want the edges to cut the m -gon into triangles.) Any triangulated polygon has a vertex with valency two, not counting multiplicities. However this vertex has valency 10 if we count

multiplicities. Thus there is an edge with multiplicity at least 5. This gives us a copy of S^2 inside D , up to rotation of the copies of S . The result now follows from the quadratic relation and the induction hypothesis. \square

Lemma 4.3. \mathcal{P}_{10} is spanned by Temperley-Lieb diagrams and S .

Proof. Suppose D is a diagram in \mathcal{P}_{10}^0 . We must show that D is a linear combination of Temperley-Lieb diagrams and S . We use induction on the number of copies of S in D .

By the braiding relation, we can assume the copies of S lie in a row at the top of D , and all strands of D lie entirely below the height of the tops of the copies of S . As in the proof of the previous lemma, we can assume that every strand connects two distinct copies of S , or has at least one endpoint on the bottom edge of D .

Suppose there is a strand that connects a non-adjacent pair of copies of S . Between these copies of S there must exist a copy of S that is connected only to its two adjacent copies. It must be connected to at least one of these by at least 5 strands. The result now follows from the quadratic relation and induction on the number of copies of S .

Now suppose every strand either connects adjacent copies of S or has at least one endpoint on the bottom edge of D . If there is exactly one copy of S in D then we have $D = S$, and we are done. Suppose D contains at least two copies of S . Either the leftmost or rightmost copy of S is connected to the bottom of D by at most 5 strands. This copy of S is connected to its only adjacent copy of S by at least 5 strands. The result now follows from the quadratic relation and induction on the number of copies of S . \square

Definition. Suppose $X \in \mathcal{P}_n^0$ for some $n < 29$. Then X is a *morphism* from P_n to the empty diagram if $XP_n = X$, or equivalently, if $XY = 0$ for every cup $Y \in \mathcal{P}_{n-2}^n$.

Lemma 4.4. If $n < 29$ and $n \notin \{0, 10, 18, 28\}$ then every morphism from P_n to the empty diagram is zero.

Proof. First note that \mathcal{P}_n is zero for odd values of n . Thus we can assume $n = 2k$. We have the following identities, shown in the case $k = 1$.

$$\boxed{X} = \boxed{X} = \boxed{X} = q^{2k(k+1)} \boxed{X}$$

The first equality is an isotopy. The second follows from the braiding relation. To prove the third, resolve each crossing and eliminate all terms that contain a cup attached to X . There are $2k$ instances of a strand crossing itself. Each of these contributes a factor of $iq^{3/2}$, by Reidemeister one. There are $2k(2k-1)$ instances of two distinct strands crossing. Each of these contributes a factor of $iq^{1/2}$.

Thus we have $X = q^{2k(k+1)}X$. If X is non-zero then $q^{2k(k+1)} = 1$, so $2k(k+1)$ is a multiple of 60. The result now follows from simple case checking. \square

Lemma 4.5. Φ is injective on \mathcal{P}_n for $n \leq 16$.

Proof. The case $n = 0$ follows from Lemma 4.2, and the case $n = 10$ follows from Lemma 4.3. Suppose $X \in \mathcal{P}_n^0$ is in the kernel of Φ , where $n \leq 16$ and $n \notin \{0, 10\}$. If Y is a cup then XY is in the kernel of Φ , so $XY = 0$ by induction on n . Thus

X is a morphism from P_n to the empty diagram. The result now follows from Lemma 4.4. \square

We are now ready to prove Theorem 4.1. Suppose $X \in \mathcal{P}_n$ is in the kernel of Φ . We must show $X = 0$. We can assume $n > 16$.

Write X as an element of \mathcal{P}_7^{n-7} . The relation in Lemma 3.3 also holds in \mathcal{P} , by Lemma 4.5. Thus X is a linear combination of diagrams of the form XAB , where $A \in \mathcal{P}_m^7$ and $B \in \mathcal{P}_7^n$ for some $m > 7$. For any such A , XA is in the kernel of Φ , so $XA = 0$ by induction on n . Thus $X = 0$. This completes the proof that Φ is an isomorphism.

5. A BASIS

We now define a set of diagrams that will form a basis for \mathcal{P}_n .

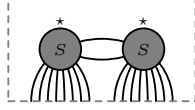
Definition. Let \mathcal{B}_n be the set of diagrams of the form XY , where X is one of the four diagrams shown in Figure refig:XSs, and Y is a JW-reduced Temperley-Lieb diagram.

The aim of this section is to prove the following.

Theorem 5.1. \mathcal{B}_n is a basis for \mathcal{P}_n .

Before we prove Theorem 5.1, we establish some additional consequences of the relations on \mathcal{P} .

Lemma 5.2. *The diagram*



is a linear combination of diagrams that have at most one copy of S .

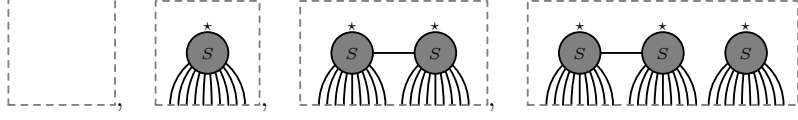
Proof. Let $\text{Join}_2(S, S)$ denote the above diagram. Consider $\text{Join}_2(S, S)P_{16}$. On the one hand, this is zero by Lemma 4.4. On the other hand, we can write P_{16} as a linear combination of Temperley-Lieb diagrams. The identity diagram occurs with coefficient one. Every other term contains a cup. This cup either connects the two copies of S or gives zero. Thus $\text{Join}_2(S, S)$ is equal to a linear combination of diagrams that contain a copy of $\text{Join}_3(S, S)$.

Now apply the same argument to $\text{Join}_3(S, S)P_{14}$, and then to $\text{Join}_4(S, S)P_{12}$. Thus we can work our way up to a linear combination of diagrams in which the two copies of S are connected by five strands. Now apply the quadratic relation to obtain a linear combination of diagrams that have at most one copy of S . \square

Lemma 5.3. *If $m \in \{0, 10, 18, 28\}$ then every morphism from P_m to the empty diagram is a scalar multiple of XP_m , where X is one of the four diagrams shown in Figure 2.*

Proof. Suppose D is a diagram in \mathcal{P}_m^0 . We must show that $DP_m = XP_m$ where X is one of the four diagrams shown in Figure 2.

We use induction on the number of copies of S in D . As in the proof of Lemma 4.3, we can assume the copies of S lie in a row at the top of D , and all strands of D lie entirely below the height of the tops of the copies of S . If there is a strand with both endpoints on the same copy of S then $D = 0$. If there is a

FIGURE 2. Possible values for X

strand with both endpoints on the bottom edge of D then $DP_m = 0$. Thus we can assume every strand either connects two copies of S , or connects a copy of S to the bottom edge of D .

By Lemma 5.2, we can assume that any pair of copies of S is connected by at most one strand. If $m \in \{0, 10, 18\}$ then the only possible values of D are as shown in Figure 2. If $m = 28$ then there is one other possibility, namely that D has three copies of S , but the second and third copy are connected by a strand, instead of the first and second. Using the braiding relation, we can bring the leftmost copy of S to the right. Thus $D = D'\beta$ where D' is the fourth diagram in Figure 2, and β is a braid. Then $D'\beta P_{28}$ is a scalar multiple of $D'P_{28}$. \square

Lemma 5.4. *Any diagram $X \in \mathcal{P}_n^0$ is a linear combination of diagrams that either contain a cap or are one of the four diagrams in Figure 2.*

Proof. Suppose $n \geq 29$. Let $P_{29} \otimes \text{id}_{n-29}$ denote the Jones-Wenzl idempotent with extra vertical strands if necessary to bring the total up to n . Consider $X(P_{29} \otimes \text{id}_{n-29})$. This is zero by Lemma 3.4. We can use this to write X as a linear combination of diagrams that contain a cap.

Now suppose $n \leq 29$ but $n \notin \{0, 10, 18, 28\}$. Consider XP_n . On the one hand, this is zero by Lemma 4.4. On the other hand, we can write P_n as a linear combination of Temperley-Lieb diagrams. We can use this to write X as a linear combination of diagrams that contain a cap.

Now suppose $n \in \{0, 10, 18, 28\}$. Consider XP_n . On the one hand, by Lemma 5.3, XP_n is a scalar multiple of DP_n , where D is one of the four diagrams in Figure 2. On the other hand, we can write P_n as a linear combination of Temperley-Lieb diagrams. Thus X is a scalar multiple of D , modulo terms that contain a cap. \square

Lemma 5.5. *Any Temperley-Lieb diagram Y in \mathcal{P}_n^m is a linear combination of diagrams that either contain a cup or are JW-reduced.*

Proof. Let x_1, \dots, x_n be the endpoints at the bottom of Y , in order from left to right. Let a_i be the number of strands that have one endpoint at or to the left of x_i , and the other endpoint to the right of x_i or at the top of Y . Let $a_0 = 0$. Call (a_0, \dots, a_n) the *sequence corresponding to Y* . This sequence satisfies $a_{i+1} = a_i \pm 1$.

Suppose $a_i \geq 29$ for some i . Then $a_i = 29$ for some i .

Let L be a vertical line between x_i and x_{i+1} . Assume all the endpoints at the top of Y are to the right of L , and L intersects the strands of Y in exactly 29 places.

Let Y' be the result of inserting a sideways copy of P_{29} into L . This is zero by Lemma 3.4. We can use this to write Y as a linear combination of diagrams obtained by inserting non-identity Temperley-Lieb diagrams into L . Each diagram in this linear combination either contains a cup, or has a corresponding sequence that is smaller than (a_0, \dots, a_n) in lexicographic order.

This process must terminate with a linear combination of Temperley-Lieb diagrams of the desired form. \square

Lemma 5.6. \mathcal{B}_n spans \mathcal{P}_n .

Proof. Suppose D is a diagram in \mathcal{P}_n . Draw D in a rectangle, with all endpoints on the bottom edge.

As in the proof of Lemma 4.3, we can assume the copies of S lie in a row at the top of D , and every strand either connects adjacent copies of S or connects a copy of S to the bottom edge of D . Let m be the number of strands that connect a copy of S to the bottom edge of D . We proceed by induction on m .

Write D in the form XY , where X is a diagram in \mathcal{P}_m^0 and Y is a Temperley-Lieb diagram in \mathcal{P}_n^m . Now apply Lemma 5.4 to X and Lemma 5.5 to Y . Thus XY is a linear combination of diagrams of the form $X'Y'$, where X either contains a cap or is one of the four diagrams in Figure 2, and Y either contains a cup or is JW-reduced.

If X' contains a cap or Y' contains a cup then $X'Y'$ has a smaller value of m . If X' does not contain a cap and Y' does not contain a cup then $X'Y'$ is an element of \mathcal{B}_n . \square

By Lemma 3.5, the dimension of \mathcal{P}_n is the number of elements of \mathcal{B}_n . This completes the proof the \mathcal{B}_n is a basis for \mathcal{P}_n .

6. THE OTHER ADE PLANAR ALGEBRAS

We now consider the subfactor planar algebras of types A_N , D_{2N} and E_6 . Each of these has a presentation and a basis similar to those we gave for E_8 , and by similar arguments.

The case of A_N is already well understood. In the case of D_{2N} , a presentation and a basis were given in [MPS08]. However our methods give a different point of view. The basis elements in [MPS08] are complicated linear combinations of diagrams built out of minimal idempotents, whereas each of our basis elements is a single diagram. This is not to say our basis is better. Indeed, the apparently more complicated basis is actually more natural from a purely algebraic point of view.

Our presentations for the planar algebras are as follows.

The subfactor planar algebra with principal graph A_N is the planar algebra with no generators and the defining relations

- $\bigcirc = [2] \bigcirc$.
- $P_N = 0$,

where $q = e^{i\pi/(N+1)}$.

The subfactor planar algebra with principal graph D_{2N} is the planar algebra \mathcal{P} with a single generator $S \in \mathcal{P}_{4N-4}$ and defining relations

- $\bigcirc = [2] \bigcirc$.
- $\rho(S) = \sqrt{-1}S$.
- $\tau(S) = 0$.
- $S^2 = P_{2N-2}$.
- $P_{4N-3} = 0$,

where $q = e^{i\pi/(4N-2)}$. Note that we cannot use a defining relation $\hat{S}P_{4N-2} = 0$, since P_{4N-2} is not defined for the above value of q .

The subfactor planar algebra with principal graph E_6 is the planar algebra \mathcal{P} with a single generator $S \in \mathcal{P}_6$ and the defining relations

- $\bigcirc = [2] \bigcirc$,
- $\rho(S) = e^{4i\pi/3} S$,
- $\tau(S) = 0$,
- $S^2 = S + [2]^2 [3] P_3$,
- $\hat{S} P_8 = 0$,

where $q = e^{i\pi/12}$.

Each of these planar algebras satisfies some kind of “braiding relation”. In the A_N case, Reidemeister moves two and three say you can drag a strand over or under any part of a diagram. In the D_{2N} case, [MPS08] prove you can drag a strand over any part of a diagram, and you can drag a strand under any part of the diagram up to a possible change of sign. In the E_6 and E_8 cases, you can drag a strand over any part of a diagram, but you cannot drag a strand under a generator, even up to sign.

In the definition of JW-reduced, we must replace the number 28 with the number k such that $P_{k+1} = 0$. In the E_6 case, $P_{11} = 0$.

Each of these planar algebras has a basis similar to the one defined in Section 5. The basis elements are of the form XY , where X is one of a short list of possibilities, and Y is a JW-reduced Temperley-Lieb diagram. In the case of A_N , X is simply the empty diagram. In the cases D_{2N} and E_6 , X is either the empty diagram or S , with all strands pointing down.

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